COMMENTS

Comments are short papers which criticize or correct papers of other authors previously published in the **Physical Review.** *Each Comment should state clearly to which paper it refers and must be accompanied by a brief abstract. The same publication schedule as for regular articles is followed, and page proofs are sent to authors.*

Reply to "Absorbing boundary conditions for inertial random processes"

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We show that the Fokker-Planck equation used in our study of the first-passage time distribution and of survival probabilities [Phys. Rev. E **48**, 2397 (1993)] for a particle moving in the field of random correlated noise describes a well defined first-order stochastic process, which is equivalent to the original second-order Langevin equation. This suggests naturally the use of the familiar first-passage boundary condition for first-order processes as done in our paper, and shows that the criticism of our boundary conditions by Masoliver *et al.* [Phys. Rev. E (to be published)] appears to be unfounded. On the other hand, the nonuniqueness of the Fokker-Planck equation for second-order random processes demonstrated by Drory renders the study of first-passage times for such processes by means of a Fokker-Planck approach unreliable. [S1063-651X(97)02302-7]

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In our previous paper [1], we studied first-passage times and survival probabilities for a particle subjected to a random acceleration described by the Langevin equation

$$\ddot{x}=f(t), \tag{1}$$

where $f(t)$ is a random Gaussian noise with zero mean assumed to have an Ornstein-Uhlenbeck correlation (with τ the correlation time)

$$\langle f(t)f(t')\rangle=\frac{f_0^2}{2\tau}\exp\left(-\frac{|t-t'|}{\tau}\right). \tag{2}$$

The analysis in Ref. [1] is based on the Fokker-Planck (diffusion) equation for the marginal distribution of the position $p(x,t)$ (which was actually derived for any form of Gaussian noise). Here, we rederive this equation in a way that explicates its relation to a first-order stochastic process. This then naturally suggests using the familiar first-passage time boundary condition for one-variable processes [2], as done in Ref. [1].

For a particle at rest at $t=0$, Eq. (1) is equivalent to the first-order stochastic equation

$$\dot{x}=\int_0^t dt' f(t'). \tag{3}$$

The probability density of displacement x is defined by

$$p(x,t)=\langle\delta[x-x(t)]\rangle, \tag{4}$$

where $x(t)$ is the solution of Eq. (3) and the brackets denote averaging over the noise. The Fokker-Planck equation for $p(x,t)$ is obtained by differentiating both sides of Eq. (4) with respect to time, i.e.,

$$\frac{\partial p(x,t)}{\partial t}=-\frac{\partial}{\partial x}\left\langle\frac{dx(t)}{dt}\,\delta[x-x(t)]\right\rangle, \tag{5}$$

and then inserting Eq. (3) and performing the average over the Gaussian driving noise Eq. (2). This readily yields the diffusion equation of Ref. [1], namely,

$$\frac{\partial p(x,t)}{\partial t}=\tfrac{1}{2}D(t)\,\frac{\partial^2 p(x,t)}{\partial x^2}, \tag{6}$$

with

$$D(t)=f_0^2 t[t-\tau(1-e^{-t/\tau})]. \tag{7}$$

Having demonstrated the equivalence of Eqs. (6) and (7) to the one-variable (or first-order) process (3) it is then natural to define first-passage times at boundaries $x=\pm\xi$ by the familiar absorbing conditions

$$p(\pm\xi,t)=0, \tag{8}$$

which have been discussed long ago for such processes [2]. These conditions used in Ref. [1] express the first-passage distribution from those realizations in which a particle starting from a position $-\xi<x_0<\xi$ at $t=0$ has never reached the boundaries during time t.

In their Comment [3], Masoliver *et al.* view Eq. (1) as describing a two-dimensional or second-order random process (x,v) obeying the coupled first-order equations

$$\dot{x}=v, \tag{9a}$$

$$\dot{v}=f(t), \tag{9b}$$

which are equivalent to (1). The statistical properties of the particle are then described by the joint distribution of position and velocity $p(x,v,t)$. The first-passage time boundary conditions at $x=\pm\xi$, which are generally used for such a two-variable process, are

$$p(-\xi,v>0,t)=0, \quad p(\xi,v<0,t)=0. \tag{10}$$

As emphasized by Masoliver *et al.* [3], these boundary conditions for first-passage times differ from the conditions (8). Equations (8) and (10) correspond to two different phenomenological definitions of distributions of the first-passage times at boundaries $\pm\xi$. In any case, in view of the general acceptance of the conditions (8) for first-passage times (and survival probabilities) for first-order processes, we are entitled to use them in connection with the Fokker-Planck equation (6) describing the first-order inertial process (3). In this sense, the criticism of our analysis on the basis of the use of improper boundary conditions by Masoliver *et al.* [3] appears to be unfounded.

These authors [3] present an interesting comparison of results for mean first-passage times obtained from our treatment [1] with those obtained by Masoliver and Porra [4] from the ordinary Fokker-Planck equation for $p(x,v,t)$, with the absorbing conditions (10) [5], in the case of white noise ($\tau=0$). This comparison reveals a good qualitative agreement between mean exit times out of a spatial interval of width L as a function of the initial position of the particle within that interval.

However, as shown recently by Drory, a fundamental ambiguity is plaguing first-passage time studies based on a Fokker-Planck equation for the joint distribution $p(x,v,t)$ and on the boundary conditions (10). This is due to the existence of two different Fokker-Planck equations that are both equivalent to Eq. (1) and have a common solution for the distribution $p(x,v,t)$ of the random process (1) in the absence of boundaries [6]. Indeed, in the presence of boundaries described by, e.g., the boundary conditions (10), one expects the solutions of the two Fokker-Planck equations to be no longer the same and thus to yield nonunique results for first-passage times.

As shown elsewhere [7], the two different Fokker-Planck equations correspond to two different but equivalent ways of writing Eq. (1) in terms of first-order differential equations, namely Eqs. (9), on the one hand, and the equations $\dot{x}=\int_0^t dt' f(t')$ and $\dot{v}=f(t)$, on the other hand. For illustration, we discuss the situation for Gaussian white-noise forces (i.e., $\tau=0$) studied by Masoliver and co-workers [3,4]. The equation for $p\equiv p(x,v,t)$ obtained from Eq. (9) [8] reduces then to the ordinary Fokker-Planck equation

$$\frac{\partial p}{\partial t}=-v\,\frac{\partial p}{\partial x}+\alpha\,\frac{\partial^2 p}{\partial v^2}, \tag{11}$$

with time-independent coefficients ($\alpha=f_0^2/2$). On the other hand, for $\tau=0$, the new Fokker-Plancklike equation [6,7], which follows from the preceding alternative first-order equations, becomes

$$\frac{\partial p}{\partial t}=\alpha\left(t^2\,\frac{\partial^2 p}{\partial x^2}+2t\,\frac{\partial^2 p}{\partial x\,\partial v}+\frac{\partial^2 p}{\partial v^2}\right), \tag{12}$$

for a particle initially at rest at $x=0$. Equations (11) and (12) have a common solution [6],

$$p(x,v,t)=\frac{\sqrt{3}}{2\pi\alpha t^3}\exp\left[-\frac{3}{\alpha t^3}\left(x^2-xvt+\frac{v^2t^2}{3}\right)\right], \tag{13}$$

describing the statistics of the motion in free space (i.e., without boundaries) defined by the Langevin equation (1). On the other hand, the solutions of Eqs. (11) and (12) in a finite interval $(-\xi,\xi)$, with the boundary conditions (10) at $x=\pm\xi$, are expected to be different, as mentioned above. From this, it follows that the mean exit time obtained by Masoliver and co-workers [3,4] from an exact solution based on Eq. (11) with the boundary conditions (10) is not unique (or unambiguous), since a different result is expected by starting from Eq. (12) with the same boundary conditions. We note, incidentally, that while Eq. (11) leads to a closed second-order partial differential equation for the mean exit time [5,4] the corresponding differential equation obtained from Eq. (12) relates the mean exit time to the second and third moments of the exit time distribution.

Finally, we emphasize that, in contrast to the Fokker-Planck equation for the distribution $p(x,v,t)$, Eqs. (6) and (7) for the marginal distribution $p(x,t)$ are unique, being related to the uniquely defined first-order process (3) for displacements x. Our results for first-passage times and survival probabilities in Ref. [1] are thus unambiguous.

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[2] See, e.g., R. L. Stratonovich, *Topics in the Theory of Random Noise* (Gordon and Breach, New York, 1967), Vol. I; H. Risken, *The Fokker-Planck Equation* (Springer, Berlin, 1989).

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